Sorting Oriented Interval-Valued Numbers with Fuzzy Measurement

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Abstract

Interval is a natural way to represent a quantity when we can not define its exact value. However comparing two intervals is not easy especially when they overlap. In order to cope with this problem, we propose a fuzzy estimation measurement. In this paper we focus on examining the so-called Oriented Interval-Valued Numbers (IVN) and show how to estimate IVN knowing the directions (the starting and finishing points) of the intervals. We build a namely Direction Oriented Comparison (DOC) method. The proposed comparison measurement allows us to use the available additional information to provide estimation with different degrees. We also show how to apply DOC to compare IVNs which can be built from oriented IVNs. Finally a practical extended DOC is provided to sort a set of IVNs of the given type.

1. Introduction: Back to the Past

The intervals are derived from many practical application problems, when instead of knowing the precise values *x* of some quantity X we know only the interval $[\underline{x}, x]$, in which X ranges. Since the comparison of two values or quantities is the basic and most frequently used step in optimization, comparison of interval plays important role in interval computation development.

Reviewing the past works [1-21], we can see the two following features. First, the definitions of interval are different and vary from a crust set, a fuzzy number, a fuzzy interval, a probabilistic interval, a real number Xo with some possible error, to a range of a variable and therefore it distinguishes the problems of comparing intervals. Second, there is a big distance between the complex theoretical estimation methods achieved [5-7] and the simple but inaccurate practical interval comparisons used in applications [8,9,20].

In the interval study within this paper we consider the interval $[\underline{x}, x]$ as the range of a variable X. That means X can have one precise value x in $[\underline{x}, x]$ and we call X (not the interval) an interval-valued number (IVN). The interval then just means the range of the IVN.

From our point of view, the IVNs can be classified based on the type and the amount of the additional information it has about the value *x* as follows: (i) IVN without any additional information, all we know are *a* and *b*; (ii) IVN with “a little” additional information such as the directions (starting and oriented points); (iii) IVN with “more” additional information such as some statistical values received in the past; (iv) IVN with “adequate” information such as the distribution form of X in [a,b], then we can consider X as a fuzzy number with some membership function; (v) IVN with “complete” information when we know the function or rule by which X receives its value in [a,b].

This paper addressed the IVN in (ii) class. The additional information is given in form of starting and finishing points or the direction. The *goal* of our study is to build *specialized* and *practical* comparison schemes focusing on simplicity and acceptable accuracy.

In order to compare IVNs of the given type, we propose a fuzzy measurement by adding the degrees to the traditional estimation. The comparison then is characterized and carried out with a unique but flexible measurement unit based on the built-in comparison degrees.

The next section presents the fuzzy measurement approach. Section 3 describes the features of oriented INV. The *Direction Oriented Comparison (*DOC) method for pairs is proposed in section 4. Then, in section 5, we extend DOC into the practical scheme for IVNs based on oriented ones and for sorting a set of IVNs. Finally section 6 is devoted to conclusion and further research.

2. Fuzzy Comparison Measurement

2.1 Related Work

With emphasis on the measurements used in interval comparison, we can see several approaches in building the estimation systems as following :

(1). Use simple estimation {*definite state*: less, more for non-overlapped intervals; *indefinite state*: for overlapped intervals} as in [8,9];

(2). Use traditional estimations {<,=,>} as for precise numbers with assumption that an interval $[\underline{x}, x]$ can be represented by $\underline{x}$ (pessimistic approach), x (optimistic approach) or $(\underline{x}+x)/2$ (average approach) [13] ;

(3). Use abstract estimation with words such as building the set of possible order relations between pairs of intervals {before,meets,overlaps, starts, during, finishes,equal}[1,4];

(4). Use complex probabilistic estimations, for instance by the four criteria PD, PSD, ND, and NSD for ranking the intervals as in [5,6].

(5).Use some intermediate function *f* on each interval with different criteria such as Hurwicz ones [11] and then sort the intervals.

The review also showed us that the most problematic cases take place when the intervals X1 and X2 are overlapped. Then any rigid estimation in (2) can not be always true, while estimations in (1)&(3) are not sufficient for ranking intervals of the similar types. The approach by (5) sounds good however sometimes to find an appropriate function *f* is even more problematic. Only (4) provided adequate theoretical basis for estimations IVNs for intervals in general form, however the comparison schemes with the four fuzzy relations on the indices [6,7] were complicated and therefore are difficult to be applied in practice.

2.2 Fuzzy Measurement Approach

Looking for a *practical* approach to comparing IVN of the (ii) type, we continue the idea proposed in [18]. The key idea is to extract the traditional estimation measurement {<,=,>} with some degree D.

We propose to represent the comparison result in form of X1 >($D^{X1,X2}$) X2 for IVNs and measure $D^{X1,X2}$ instead of just define if X1>X2, X1<X2, X1=X2 or not as for the precise numbers. By adding the degrees to the estimation in (2) we indeed fuzzify the result of the comparison. To suit the uncertain situation when the intervals are overlapped we propose "*to fight fire with fire*". That is to use a fuzzy (uncertain in some sense) measurement to describe the irregular (inconsistent) relationship of the two IVN.

This fuzzy comparison approach has been experimentally applied to the (i) IVN which are given without any additional information in [17,18]. Several fuzzy comparison schemes have been proposed:

In Extended Comparison [18] we add three degrees "precise", "relative", and "probably" to the classical signs {<,=,>}.
In Multi-Level Comparison [18] each of the signs {<,=,>} is extended to n levels and we have, for instance, X1 >(n) X2 where n is a positive integer.
In Continuous Comparison [18] we use the notation X1 >(G) X2, where G is a real number and 0<G<1 when the intervals are overlapped, otherwise G=1 or G=0.
In Complete Continuous Comparison [17] we extend the computation of degree G for non-overlapped cases.

However, regardless what kind of techniques we use, any designed comparison rule or scheme for IVNs of type (i) can not provide an always-true answer about the relationship between the two intervals. Because when there is no any additional information such an answer does not exit, unless we have to assume that the IVNs have normal distribution form.

Fortunately, as practice shows, in many cases we can have some additional information about how the true value x of the IVN may be distributed in the interval [$\underline{x}$, $\overline{x}$]. Then the probabilistic estimation can be applied and the estimation measurement should be specialized on the type of the available additional information.

For the IVNs of the type (iii) we have proposed a specialized Statistic-Oriented Estimation method based on a pseudo-distribution measurement [19]. We have built three alternative comparison schemes namely:

Value-Based Scheme for IVN with chaotic pseudo-distribution form;
Frequency-Based Scheme for IVN with one-peak pseudo-distribution form;
Density-Based Scheme for IVN with non-peak pseudo-distribution form.

To fulfill the lack of work on IVN of the (ii) type and to show the practical effectiveness of the proposed fuzzy comparison measurement, we devote this paper to solving the problem of comparing the so-called time-oriented IVNs - one of the most popular types in real-time systems, which will be described in the next section.

3. Oriented IVNs

In order to examine the value x that an IVN X can receive in its range $[\underline{x}, \overline{x}]$, we suppose that $x = F(t)$, where F is some function of a virtual time (or space, or volume) variable t. We also assume that there is a virtual generator of X which generates its value x virtually by time, from a beginning time moment To with Xo (starting point) to some finishing time moment Te with Xe (finishing point) as shown inFig1.

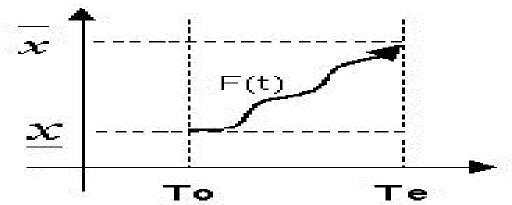


Figure 1

Definition 1: (Time-dependent IVN)

Given an IVN X with its range $[\underline{x}, \overline{x}]$. X is called *time-dependent* if there is some function F(t) of some real variable (can be time, volume, or space) that for each $t \in [To,Te]$: $\exists!\ x = F(t) \in [\underline{x}, \overline{x}]$.

Definition 2: **(Oriented IVN)**

Given a time dependent IVN X with its range $[\underline{x}, \overline{x}]$. X is called *Oriented* if $\forall$ t1≠t2 and t1,t2∈ [To,Te]: If t2>t1 then: |Xe-x2| < |Xe-x1| or |x2-Xo| > |x1-Xo|, where Xo=F(To); Xe=F(Te); x1=F(t1); and x2=F(t2). (Fig.2)

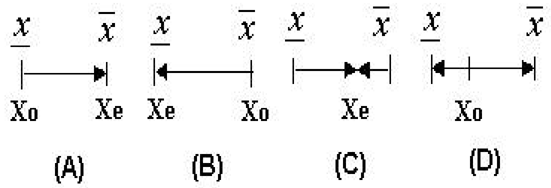


Figure 2

Definition 3: **(Monotonic IVN)**

Given an oriented IVN X with its range $[\underline{x}, \overline{x}]$. X is called *Monotonic* if with t1,t2∈ [To,Te]:

$\forall$ t2>t1: x1=F(t1)> x2=F(t2); Then Xe= $\underline{x}$ and Xo= $\overline{x}$ **or**

$\forall$ t2>t1: x1=F(t1)< x2=F(t2); Then Xe= $\overline{x}$ and Xo= $\underline{x}$

The monotonic IVNs are the two special cases A and B of the four cases A,B,C,D of oriented IVN (Fig2).

4. Direction-Oriented Comparison

4.1 Problem Statement

Given: Two monotonic IVN: X and Y with their range $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$ and $\overline{x} \geq \underline{x}$ and $\overline{y} \geq \underline{y}$. The starting and finishing points Xo and Xe for X; Yo and Ye for Y. Suppose that we can represent the comparison measurement of X and Y by one of the two following notations $X>(D^{x,y})Y$ or $X>>((H^{x,y}))Y$, where $D^{x,y}$ is called the Degree of more and $H^{x,y}$ is called the Level of more.

Questions:

How can we build the degrees $D^{x,y}$ and $H^{x,y}$ so that:

+) $D^{x,y}$ can distinguish the overlapping cases and represent degrees of our belief that X is more than Y in those cases.

+) $H^{x,y}$ can represent the level of how much X>Y (X<Y) in non-overlapping cases and distinguish the case when X=Y.

Notes: In further study when we write X Ω Y where Ω={<,=,>} that means $\forall x \in [\underline{x}, \overline{x}]$ and $\forall y \in [\underline{y}, \overline{y}]$ we have x Ω y. Otherwise we use the new mentioned signs with the degree D or level H.

4.2 Movement Model for Measuring

In order to examine the possibility that x of X is more than y of Y we propose to build the measurement model as the follows.

Suppose that $\Delta x = \overline{x} - \underline{x}$; $\Delta y = \overline{y} - \underline{y}$; Δt = Te-To; Then, we can investigate the possible values x of X in the range $[\underline{x}, \overline{x}]$ and y of Y in $[\underline{y}, \overline{y}]$ by simulating a virtual movement of x from starting points Xo and Yo to the finishing points Xe and Ye with velocities (Fig.3):

$Vx = \Delta x / \Delta t$ (1)
$Vy = \Delta y / \Delta t$ (2)

Next, at a current time moment t, the current positions of x in $[\underline{x}, \overline{x}]$ and y in $[\underline{y}, \overline{y}]$ can be defined as:

$x = Xo + (-1)^{dx} \times Vx \times t$;
$y = Yo + (-1)^{dy} \times Vy \times t$;

where dx (dy) represents the oriented direction of X (Y):

dx=0 (dy=0) if if we have $x2 \geq x1 (y2 \geq y1)$ when $t2 \geq t1$.

dx = 1 (dy=1) if we have $x2 \geq x1$ $(y2 \geq y1)$ when $t1 \geq t2$.

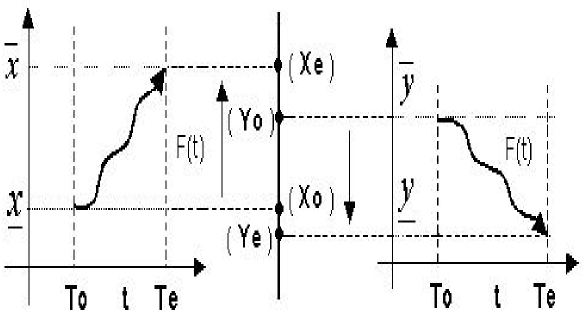


Figure 3

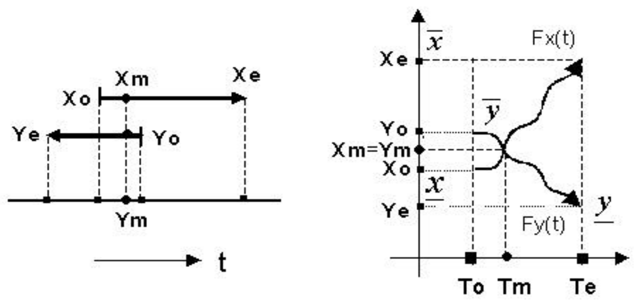
Assume that there will also be a meeting point at some time moment Tm when x=Xm=y=Ym (Fig 4). Then we have:

$Xm = Xo + (-1)^{dx} \times Vx \times Tm$;
$Ym = Yo + (-1)^{dx} \times Vy \times Tm$;
Xm=Ym;

That means:

$Xo + (-1)^{dx} \times Vx \times Tm = Yo + (-1)^{dy} \times Vy \times Tm$;

Suppose that $\Delta t \neq 0$, then :



$\frac{Tm}{\Delta t}$ ($(-1)^{dy} \times \Delta y - (-1)^{dx} \times \Delta x$) = Xo-Yo (1)

Figure 4

Note 1:

If $(-1)^{dy} \times \Delta y - (-1)^{dx} \times \Delta x = 0$ or $(-1)^{dy} \times \Delta y = (-1)^{dx} \times \Delta x$, then $\Delta y = \Delta x$ and dx = dy.

if $Xo \neq Yo$ there is no solution for Tm in (1). That means there is no meeting point. In this case X>Y if Xo>Yo and X<Y if Xo<Yo.

if Xo = Yo any real number is the solution for Tm (1). That means every point in X and Y is the meeting point. In this case X=Y.

Note 2:

If $(-1)^{dx} \times \Delta y - (-1)^{dy} \times \Delta x \neq 0$ (then $Xo \neq Yo$) we have:

$$\frac{Tm}{\Delta t} = \frac{Xo - Yo}{((-1)^{dx} \times \Delta x - (-1)^{dy} \times \Delta y)} = \mathbf{K}$$

If $K > 1$ or $K < 0$ that means $Tm \notin [To, Te]$ or there is no meeting point in X and Y. In this case X>Y if Xo>Yo and X<Y if Xo<Yo.

If $0 \le K \le 1$ that means $\exists\, Tm \in [To, Te]$ or there is a meeting point in X and Y. In this case we have:

When $To \le t \le Tm$: x>y if Xo>Yo and x <y if Xo<Yo;

When $Tm \le t \le Te$: x<y if Xo>Yo and x>y if Xo<Yo;

4.3 Comparison Scheme

Using the results obtained in the previous section we can build the comparison scheme, in which the estimation (degree $D^{x,y}$) is defined based on the known directions and the four indices. Therefore we name it Direction-Oriented Comparison (DOC). It is stated as following:

Given: Two monotonic IVN: X and Y with their range $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$. The starting and finishing points Xo and Xe for X; Yo and Ye for Y. Suppose that X has direction dx and Y has dy and $\Delta x = \overline{x} - \underline{x}$; $\Delta y = \overline{y} - \underline{y}$;

Algorithm:

For Degree D

If $(-1)^{dx} \times \Delta y - (-1)^{dy} \times \Delta x \neq 0$
then :
if $0 \le K \le 1$
then We have $X>(D^{x,y})Y$ where $D^{x,y}=K$
or $Y>(D^{y,x})X$ where $D^{y,x}=1-K$
else
if Xo>Yo **then** X>Y (determined cases)
else X<Y (determined cases)
else
if $Xo \neq Yo$
then
if Xo>Yo **then** X>Y (determined cases)
else X<Y (determined cases)
else X=Y (determined cases)

For Degree H

In the determined cases when X>Y, X<Y, or X=Y we write:

X>>($H^{x,y}$)Y
Where, $H^{x,y} = (\overline{x} + \underline{x})/2 - (\overline{y} + \underline{y})/2$;
Then,
if $H^{x,y}=0$ then X=Y;
if $H^{x,y}>0$ then X>Y;
if $H^{x,y}<0$ then X<Y;
We also notice that $H^{x,y} = -H^{y,x}$;

5. Sorting IVN with Extended DOC

5.1 Extending DOC for other IVNs

<u>Definition 4:</u> (Multi-Oriented IVN)

Given an IVN X with its range $[\underline{x}, \overline{x}]$. X is called Multi-Oriented if X can be divided into definite number of monotonic IVNs. That means:

$$X = X1 \cap X2 \cap \ldots \cap Xn$$

Where X1, X2,…, Xn are monotonic and *n* is definite.

Problem 1

<u>Given:</u> ***two multi-oriented IVN: X and Y with their range***

$[\underline{x}, \overline{x}] = X = X1 \cap X2 \cap \ldots \cap XN$ and
$[\underline{y}, \overline{y}] = Y = Y1 \cap Y2 \cap \ldots \cap YM$

The starting and finishing points:

Xo(1),Xo(2),…, Xo(N) and Xe(1),Xe(2),…,Xe(N) for X;

Yo(1),Yo(2),…, Yo(M) and Ye(1),Ye(2),…,Ye(M) for Y.

Question: we have to compare the given IVNs : X and Y.

Solution

For i=1 to N do

For j=1 to M do

$$D^{X(i),Y(j)} = \frac{Xo(i)-Yo(j)}{((-1)^{dy(j)} \times \Delta y(j) - (-1)^{dx(i)} \times \Delta x(i))}$$

Then,

$$D^{x,y} = \frac{\sum_{i=1}^{N} \sum_{j=1}^{M} D^{X(i),Y(j)}}{N+M}$$

The rest of the comparison algorithm including calculating $H^{x,y}$ remains the same as in 4.3.

Comment: Since oriented IVN is the a special case of multi-oriented IVNs, we can use this extended comparison scheme for comparing oriented IVNs as well. Then the cases C and D (Fig.2) can be solved by virtually dividing into IVN of types A and B.

5.2 Sorting Oriented IVNs

Problem 2

Given: Q oriented or multi-oriented IVNs : Z1, Z2,…,ZQ with their ranges: [$\underline{z1}$, $\overline{z1}$],[$\underline{z2}$, $\overline{z2}$], …, [$\underline{zQ}$, $\overline{zQ}$] With the starting and finishing points:

Zo(1), Zo(2),…, Zo(N) and Ze(1), Ze(2),…, Ze(N);

Question: Sort the Q given IVNs in increasing order.

Solution

1. Take the first Z1 as the basic for comparison and then compute the degrees $H^{Z(i),Z(1)}$, where i=2..Q for Q-1 remain IVNs.
2. Sort all of Q IVNs based on their $H^{Z(i),Z(1)}$, where i=1..Q. Suppose we have the following order:

$$Z^{1*}, Z^{2*},\ldots,Z^{Q*}$$

Where $H^{Z(i)*,Z(1)} \leq H^{Z(i+1)*,Z(1)}$ with i=1..Q-1 ;

3. B=Z1* ; q=1; Set the set S empty S=∅;
4. For i=q to Q compute $D^{Z(i)*, B}$
5. Take IVNs with $0< D^{Z(i)*, B} <1$ into a buffer set Sb. Suppose we have Nq such IVNs.
6. Sort them in Sb by $D^{Z(i)*, B}$
7. Add Sb to S and set B=the last IVN in Sb; q=q+Nq;
8. Check if q<Q go to 4 otherwise finish.

6. Conclusion: Toward the Future

Many existing optimization algorithms with assumptions about data availability cannot be applied in practice because of the lack of information in the real world. In many cases, the data disharmony is due to the fact that data assumed available as exact numbers is only available as interval-valued numbers (IVN).

That is why the interest in interval computation has been raised for the last few years. However while numerous papers focus on interval arithmetic there are not as much on interval comparison [10,12].

In this paper we have built a namely Direction-Oriented-Comparison (DOC) method for pair comparison of monotonic INV and then have shown how DOC can be extended to sort the set of oriented IVNs as well as of multi-oriented IVNs.

The proposed comparison method is based on using the available additional information such as starting and finishing points or direction of the intervals to fuzzify the comparison measurement and therefore allows us to give the estimation in more details.

This work is one of the steps in the work series [17,18,19] on interval comparison, whose goal is to build practical tools and mechanisms for applying optimization algorithms that work with exact numbers to work with intervals. Further research will be continued by combining the developed tools into a unique but flexible comparison system that can adapt to different types of IVN or different types of the available additional information. After that we expect to apply this system to solve several applications [17] from which this work series was begun.

Acknowledgment

The authors would like to thank professors Peter Haddawy, Henry Prade, Didier Dubois, and Vladik Kreinovich for their valuable comments and helps in obtaining papers on interval comparison. We also thank the President of Assumption University - Rev. Bro. Prathip Martin Komolmas for his constant support. This work is done under ISL projects of Assumption University.

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