The Application of the Extended Conjugate Gradient Method on the 
One-Dimensional Energized Wave Equation

Victor O. Waziri and Sunday A. Reju*

Mathematics, Statistics & Computer Science Department, Federal University of Technology 
Minna, Nigeria
Email: dronomzawaziri@yahoo.com, sreju@nou.edu.ng

Abstract

This paper computes the optimal control and state of the one-dimensional 
Energized wave equation using the Extended Conjugate gradient Method (E CGM). We 
recalled all vital computational issues in the implementation of the ECGM algorithm on 
the one-dimensional Energized Wave equation in the paper. With these recalls, program 
codes were derived which gave various numerical optima controls and states. These 
optimal controls and states were considered as various points in thin rod as our 
computational problem is a one dimensional space wave problem.

Keywords: Extended conjugate gradient method, conjugate gradient method, 
control operator.

Introduction

We recalled the Energized wave equation as in (Odio et al. 1998) and (Waziri, 2004). 
The implementation of the ECGM algorithm on the Energized Wave equation follows the 
pattern of the Conjugate Gradient Method (CGM) developed by (Hestenes and Stiefel, 
1952). The difference between the ECGM and the CGM is in the construction of the control 
operator. The ECGM algorithm, as we have rightly stated, was developed by Ibiejugba and 
Onumanyi (1984). In the sequel, during the course of the implementation of algorithm 
computational procedures, we shall recall some fundamental useful results already obtained in 
the previous papers of this series.

Theory

Recall the unconstrained problem of the optimization Energized wave equation in 
Waziri and Reju (2006a). The unconstrained problem is here reproduced for accessibility 
and convenience:

\[
\min_{z,u} J(z,u) = \min_{z,u} \int_0^1 \int_0^1 \left[ z^2(x,t) + u^2(x,t) \right. \\
+ \mu \left[ \frac{\partial^2 z(x,t)}{\partial t^2} + \frac{\partial z(x,t)}{\partial t} \\
- \frac{\partial^2 z(x,t)}{\partial x^2} - u(x,t) \right] \right] \, dx \, dt .
\]  

The control penalized gradient is defined as 

\[
J_{u,\mu}(z,u,\mu) = 2 \left[ \int_0^1 \int_0^1 \left[ z^2(x,t) + u^2(x,t) \right] \right] + \\
\mu \left[ \frac{\partial^2 z(x,t)}{\partial t^2} + \frac{\partial z(x,t)}{\partial t} \cdot \frac{\partial^2 z(x,t)}{\partial x^2} - u(x,t) \right] \, dt \, dx .
\]  

The state penalized gradient is defined as:

\[
J_{z,\mu}(z,u,\mu) = 2 \int_0^1 \int_0^1 z(x,t) \, dt \, dx .
\]  

From Eq. (2), rewrite Eq. (4) as

\[
p_{u,\mu}(x,t) = \int_0^1 \int_0^1 J_{u,\mu}(z,u,\mu) \, dx \, dt
\]
\[ J_w(z,u,\mu) = 2xt \left[ \int_0^1 \left[ z^2(x,t) + u^2(x,t) \right] dt \right] + \mu \left[ \frac{\partial^2 z(x,t)}{\partial t^2} + \frac{\partial^2 z(x,t)}{\partial t} - \frac{\partial^2 z(x,t)}{\partial x^2} - u(t) \right] dt dx. \] (5)

The state penalized unconstraint gradient is:

\[ P_{z,j}(x,t) = 2xt \int_0^1 \frac{d}{dx} z(x,t) dx dt. \] (6)

Recall the Hamiltonian function for which the optimal solutions in Waziri and Reju (2007) were obtained:

\[ u(x,t) = \left[ \frac{\lambda_2^3 - 2\lambda_1^2}{\lambda_2 - \lambda_1} \sum_{i=1}^{\infty} u_i(0) \sin \pi mx - \frac{1}{\lambda_2 - \lambda_1} \sum_{i=1}^{\infty} u_i(0) \sin \pi mx \lambda_2 \right] e^{\lambda_2 t} + \left[ \frac{1}{\lambda_2 - \lambda_1} \sum_{i=1}^{\infty} u_i(0) \sin \pi mx \lambda_2 \right] e^{\lambda_2 t}, \] (7)

\[ z(x,t) = \left[ \frac{\lambda_2^2 - 2\lambda_1^2}{\lambda_2 - \lambda_1} \sum_{i=1}^{\infty} u_i(0) \sin \pi mx \lambda_2 \right] e^{\lambda_2 t} + \left[ \frac{1}{\lambda_2 - \lambda_1} \sum_{i=1}^{\infty} u_i(0) \sin \pi mx \lambda_2 \right] e^{\lambda_2 t}, \] (8)

The ECGM Implementation Algorithm

In the previous section, we developed some basic computational elements that will be useful in the implementation of the ECGM computational algorithm. The format of this implementation algorithm is reproduced hereunder using the CGM pattern. Nonetheless, if compared to the nature of our problem, the following systematic ECGM is an elegant algorithm which is comparatively better than the CGM algorithm. The descent directions for the control and state are:

\[ P_{z,p} = -g_{u,p} = -(a + Bu_{p}) , \] (12)

\[ P_{z,p} = -g_{z,p} = -(a + Bu_{z,p}) . \] (13)

The \( n^{th} \) iterative optimal state and control are obtained by this set of equations:

\[ z_{i+1} = z_i + \alpha_{z,i} P_{z,i} , \] (14)

\[ u_{i+1} = u_i + \alpha_{u,i} P_{u,i} . \]

The \( n^{th} \) gradients for the state and control are obtained from this set of equations:

\[ g_{z,j+1} = g_{z,j} + \alpha_{z,j} B_{z,j} P_{z,i} , \] (15)

\[ g_{u,j+1} = g_{u,j} + \alpha_{u,j} B_{u,j} P_{u,i} . \]

The \( n^{th} \) iterative descent directions for the state and control are derived from these two equations,

\[ P_{z,j+1} = -g_{z,j+1} + \beta_{z,j} P_{z,i} , \] (16)

\[ P_{u,j+1} = -g_{u,j+1} + \beta_{u,j} P_{u,i} , \] where the various step-lengths are defined in this sequential order:
\[
\alpha_{u,j} = \frac{<g_{u,j}, g_{u,j}>}{<P_{u,j}, B_{u,j}P_{u,j}>},
\]
\[
\alpha_{z,j} = \frac{<g_{z,j}, g_{z,j}>}{<P_{z,j}, B_{z,j}P_{z,j}>},
\]
\[
\beta_{u,j} = \frac{<g_{u,j+1}, g_{u,j+1}>}{<g_{u,j}, g_{u,j}>},
\]
\[
\beta_{z,j} = \frac{<g_{z,j+1}, g_{z,j+1}>}{<g_{z,j}, g_{z,j}>}.
\]  

(17)

Before utilizing Eqs. (16) and (17) in the computational procedures, we need to construct the terms \(B_{z,j}P_{z,j}\) and \(B_{u,j}P_{u,j}\) vectors as derived from the control operator in Waziri and Reju (2006a) and from Eqs. (4) and (6). We construct the \(B_{z,j}P_{z,j}\) vector from the control operator elements \(B_{11}\) and \(B_{22}\) as obtained in Waziri and Reju (2006b) aforementioned in this sequential order:

\[
B_{z,j}P_{z,j} = 3P_{z,0}(x,t) - P_{z,0}(x,0) - 3t\frac{\partial P_{z,0}(x,0)}{\partial t} + \frac{3t^2\partial^2 P_{z,0}(x,0)}{x^2} + \frac{3t^2}{2} \frac{\partial^2 P_{z,0}(x,0)}{\partial t^2} + \frac{t^2}{2} \frac{\partial^2 P_{z,0}(x,0)}{\partial t^2} + (1 + \mu)P_{u,j}.
\]

(19)

Further simplification yields the state vector \(B_{z,j}P_{z,j}\) as follows:

\[
B_{z,j}P_{z,j} = 2 \int_{0}^{x} \int_{0}^{t} \! J_{z,0}(z_i, u_i, \mu) dx dt - \int_{0}^{x} \int_{0}^{t} \! J_{z,0}(z_i, u_i, \mu) dx dt - \int_{0}^{x} \int_{0}^{t} \! J_{z,0}(z_i, u_i, \mu) dx dt - \int_{0}^{x} \int_{0}^{t} \! J_{u,0}(z_i, u_i, \mu) dx dt - \int_{0}^{x} \int_{0}^{t} \! J_{u,0}(z_i, u_i, \mu) dx dt - \frac{3t^2}{x} z_i(x,0) + \frac{3t^2}{x^2} z_i(x,0) + \frac{t^2}{2} \int_{0}^{x} \! J_{u,0}(z_i, u_i, \mu) dx - \frac{2t^2}{3} u_i(x,0) + \frac{5t^2}{6} \int_{0}^{x} \! J_{u,0}(z_i, u_i, \mu) dx + \frac{t^4}{t^3} \int_{0}^{x} \! J_{z,0}(z_i, u_i, \mu) dx .
\]

(20)

Now, appropriately applying the analytical solutions as given in the previous section, we obtained the general state vector:

\[
B_{z,j}P_{z,j} = [2tx + \frac{x^3}{3} + \frac{2t^5}{x} \int_{0}^{x} \sum_{i=1}^{\infty} u_{i}(0) \sin \pi x dt + 2\mu(t - 2tx) \int_{0}^{\infty} \left[ \lambda_{2} - 2\lambda_{1} \sum_{i=1}^{\infty} u_{i}(0) \sin \pi x \right] e^{\lambda_{1}t} + \frac{\lambda_{1}}{\lambda_{2} - \lambda_{1}} \sum_{i=1}^{\infty} u_{i}(0) \sin \pi x e^{\lambda_{1}t} + 2\frac{\lambda_{2}}{\lambda_{2} - \lambda_{1}} \sum_{i=1}^{\infty} u_{i}(0) \sin \pi x e^{\lambda_{2}t} + 2\frac{\lambda_{2}}{\lambda_{2} - \lambda_{1}} \sum_{i=1}^{\infty} u_{i}(0) \sin \pi x e^{\lambda_{1}t} + 2\frac{\lambda_{2}}{\lambda_{2} - \lambda_{1}} \sum_{i=1}^{\infty} u_{i}(0) \sin \pi x e^{\lambda_{2}t} + 2\frac{\lambda_{2}}{\lambda_{2} - \lambda_{1}} \sum_{i=1}^{\infty} u_{i}(0) \sin \pi x e^{\lambda_{2}t} .
\]

(21)

In a characteristic constructional procedure as for the \(B_{z,j}P_{z,j}\) vector, the \(B_{u,j}P_{u,j}\) vector is obtained from \(B_{12}\) and \(B_{21}\) to give:

\[
B_{u,j}P_{u,j} = 4P_{u,j}(x,0) + \frac{x^4}{t^3} \frac{\partial P_{u,j}(x,0)}{\partial t} + \frac{x^4}{2} \frac{\partial^2 P_{u,j}(x,0)}{\partial t^2} + \frac{2}{t^3} \frac{\partial P_{u,j}(x,0)}{\partial t} + \frac{x^4}{2} \frac{\partial^2 P_{u,j}(x,0)}{\partial t^2} + \frac{\partial^2 P_{u,j}(x,0)}{\partial x^2} - \frac{\partial P_{u,j}(x,t)}{\partial t} - \frac{\partial P_{u,j}(x,t)}{\partial t} .
\]

(22)

Further simplification of Eq. (22) yields the control vector \(B_{z,j}P_{z,j}\) as follows:


\[ B_{u,d} P_{u,d} = \left[ 18t_x + \frac{3x^3}{3} + \right. \]
\[ \sum_{i=1}^{\infty} u_i(0) \sin \pi x dx + 2\mu t_x^2 \] \[ \]
\[ 4xt + 2(1 + \mu) xt \]
\[ \int_{0}^{\infty} \int_{0}^{\infty} \left[ \frac{\lambda_1, \lambda_2 - 2\lambda_1^2}{\lambda_2 - \lambda_1} \right] \]
\[ \sum_{i=1}^{\infty} u_i(0) \sin \pi x dx - \]
\[ \frac{1}{\lambda_2 - \lambda_1} \sum_{i=1}^{\infty} u_i(0) \sin \pi x \left. \right|^{e^{\lambda_1 t}} - \]
\[ \frac{\lambda_1}{\lambda_2 - \lambda_1} \sum_{i=1}^{\infty} u_i(0) \sin \pi x \left. \right|^{e^{\lambda_1 t}} - \]
\[ 2\left[ \frac{x^4 + x^2}{2} t^2 \right] \int_{0}^{\infty} \left[ \frac{\lambda_1, \lambda_2 - 2\lambda_1^2}{\lambda_2 - \lambda_1} \right] \]
\[ \sum_{i=1}^{\infty} u_i(0) \sin \pi x dx - \]
\[ \frac{\lambda_2}{\lambda_2 - \lambda_1} \sum_{i=1}^{\infty} u_i(0) \sin \pi x \left. \right|^{e^{\lambda_1 t}} + \]
\[ \frac{\lambda_2, \lambda_1}{\lambda_2 - \lambda_1} \sum_{i=1}^{\infty} u_i(0) \sin \pi x \left. \right|^{e^{\lambda_1 t}}. \] 

(23)

In conjunction with the derived penalty cost functional \( \mu(x,t) \geq 0 \) as given in Waziri and Reju (2006b), we can appropriately substitute Eqs. (21) and (23) into the derived ECGM sequential analytical computational algorithm to obtain the desirable optimal controls and states at different points for the one-dimensional optimization energized wave equation.

### The Optimal Control and State Outputs

We make the following observations in summarized tabular form under various values of the initial amplitudes and velocities at various one-dimensional points in a straight line-like rod space and tolerance (\( \varepsilon = 0.0001 \)) by the application of program codes. Tables 1 and 2, respectively, give the general summary of the optimal control and state.

#### Table 1. The optimal control outputs are given at various dimensional points.

<table>
<thead>
<tr>
<th>Point profiles</th>
<th>Optimal control ( u(x,t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-2.4838613502098x10^{-3}</td>
</tr>
<tr>
<td>10</td>
<td>-2.4857096941292x10^{-3}</td>
</tr>
<tr>
<td>20</td>
<td>-1.110649111191x10^{-3}</td>
</tr>
<tr>
<td>30</td>
<td>-1.11064417816319x10^{-3}</td>
</tr>
<tr>
<td>40</td>
<td>-1.1106424500583x10^{-3}</td>
</tr>
<tr>
<td>50</td>
<td>-1.11064164995x10^{-3}</td>
</tr>
<tr>
<td>60</td>
<td>-1.11064125665x10^{-3}</td>
</tr>
</tbody>
</table>

#### Table 2. The optimal control outputs.

<table>
<thead>
<tr>
<th>Point profiles, ( n )</th>
<th>Optimal State ( z(x,t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-6.9177714504089x10^{-7}</td>
</tr>
<tr>
<td>10</td>
<td>-3.37075632986434x10^{-10}</td>
</tr>
<tr>
<td>20</td>
<td>-9.4025669392153x10^{-12}</td>
</tr>
<tr>
<td>30</td>
<td>-1.8569500540829x10^{-13}</td>
</tr>
<tr>
<td>40</td>
<td>-5.8751944507679x10^{-13}</td>
</tr>
<tr>
<td>50</td>
<td>-2.406428727935x10^{-13}</td>
</tr>
<tr>
<td>60</td>
<td>-1.1660491311665154x10^{-12}</td>
</tr>
</tbody>
</table>

### Conclusion

The results obtained for the optimal states and controls in Tables 1 and 2 are self-revealing. We observe that as the dimensional points “\( n \)” in space increase from \( n = 2 \) and \( n = 10 \), the optimal state values are relatively stable with just a negligible local error. Between \( n = 20 \) and \( n = 60 \), the optimal control solutions are stable but the state values alternate in values. Hence, within this range we conclude that the optimal control outputs converge more rapidly than the optimal state as “\( n \)” increases in value.

The optimal solutions give the solutions of the state and control in a thin medium of a wave propagating in one-dimensional space, say a straight rod. The various optimal values for the state and control represent unique optimal outputs in various points in some given one-dimensional rod.

### References